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Exotic Smoothness, Noncommutative Geometry, and Particle Physics

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We investigate how exotic differential structures may reveal themselves in particle physics. The analysis is based on A. Connes' construction of the standard model. It is shown that, if one of the copies of the spacetime manifold is equipped with an exotic differential structure, a compact object of geometric origin may exist even if the spacetime is topologically trivial. Possible implications are discussed. An $SU(3) \otimes SU(2) \otimes U(1)$ gauge model is constructed. This model may not be realistic, but it shows what kind of physical phenomena might be expected due to the existence of exotic differential structures on the spacetime manifold.

There is no interesting topology on $\mathbf{R}^4$, the Euclidean four-dimensional space (or, to be more precise, it is topologically equivalent to a single point space). The counterintuitive results (Freedman, 1982; Donaldson, 1983; Gompf, 1983, 1993; DeMichelis and Freedman, 1992) that $\mathbf{R}^4$ may be given infinitely many exotic differential structures raised the question of their physical consequences (Brans and Randall, 1993; Brans, 1994; Sładkowski, 1996). An exotic differential structure $\hat{C}^k(M)$ on a manifold M is, by definition, a differential structure that is not diffeomorphic to the one considered as a standard one, $C^k(M)$. This means that the sets of differentiable functions are different. For example, there are functions on $\mathbf{R}^4$ that are not differentiable on some exotic $\mathbf{R}^4_\Theta$ which is homeomorphic but not diffeomorphic to $\mathbf{R}^4$. Here we would like to investigate the role that exotic differential structures on the spacetime manifold may play in particle physics. Our starting point will be A. Connes' noncommutative geometry-based construction of the standard model (Connes, 1994; Várilly and Garcia-Bondía, 1993; Chamseddine *et al.*, 1993; Sładkowski, 1994a,b). Connes managed to reformulate the standard notions of differential geometry in a pure algebraic

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way that allows one to get rid of the differentiability and continuity requirements. The notion of spacetime manifold S can be equivalently described by the (commutative) algebra $C^\infty(S)$ of smooth functions on S and can be generalized to (*a priori*) an arbitrary noncommutative involutive algebra. Fiber bundles became projective modules in this language. A properly generalized connection can describe gauge fields on these objects. This allows one to incorporate the Higgs field into the gauge field so that the correct (that is, leading to spontaneously broken gauge symmetry) form of the scalar potential is obtained. The reader is referred to Connes (1994), Várilly and Garcia-Bondía (1993), Chamseddine *et al.* (1993), and Sładkowski (1994a,b) for details.

We shall consider the algebra A:

$$A = M_1(C^\infty(S)) \oplus M_2(C^\infty(S)) \oplus M_1(\hat{C}^\infty(S)) \oplus M_3(\hat{C}^\infty(S)) \tag{1}$$

where $M_i(ring)$ denotes $i \times i$ matrices over the ring $C^\infty(S)$ or $\hat{C}^\infty(S)$. The caret denotes that the functions are smooth with respect to some nonstandard differential structure on S. The free Dirac operator has the form

$$D = \begin{pmatrix} \partial\!\!\!/ \otimes \mathrm{Id} & \gamma_5 \otimes m_{12} & \gamma_5 \otimes m_{13} & \gamma_5 \otimes m_{14} \\ \gamma_5 \otimes m_{21} & \partial\!\!\!/ \otimes \mathrm{Id} & \gamma_5 \otimes m_{23} & \gamma_5 \otimes m_{24} \\ \gamma_5 \otimes m_{31} & \gamma_5 \otimes m_{32} & \hat{\partial\!\!\!/} \otimes \mathrm{Id} & \gamma_5 \otimes m_{34} \\ \gamma_5 \otimes m_{41} & \gamma_5 \otimes m_{42} & \gamma_5 \otimes m_{43} & \hat{\partial\!\!\!/} \otimes \mathrm{Id} \end{pmatrix} \tag{2}$$

Here, as before, the caret denotes the "exoticness" of the appropriate differential structure. The parameters m_{ij} describe the fermionic mass sector. Let ρ be a (self-adjoint) one-form in $\Omega^1(A) \subset \Omega^*(A)$; here $\Omega^*(A)$ denotes the universal differential algebra of A (Connes, 1994; Várilly and Garcia-Bondía, 1993):

$$\rho = \sum_i a_i db_i, \qquad a_i, b_i \in A; \qquad \sum_i a_i b_i = 1 \tag{3}$$

We will use the following notation for an $a \in A$:

$$a = \mathrm{diag}(a^1, a^2, a^3, a^4) \tag{4}$$

with a^i belonging to the appropriate matrix algebra in (1). The physical bosonic fields are defined via the representation π in terms of (bounded) operators in the appropriate Hilbert space (Connes, 1994; Várilly and Garcia-Bondía, 1993; Chamseddine *et al.*, 1993; Sładkowski, 1994a,b):

$$\pi(a_0 da_1 \cdots a_n) = a_0[D, a_1] \cdots [D, a_n] \tag{5}$$

Standard algebraic calculations lead to

$$\pi(\rho) = \begin{pmatrix} A^1 & \gamma_5 \otimes \phi^{12} & \gamma_5 \otimes \phi^{13} & \gamma_5 \otimes \phi^{14} \\ \gamma_5 \otimes \phi^{21} & A^2 & \gamma_5 \otimes \phi^{23} & \gamma_5 \otimes \phi^{24} \\ \gamma_5 \otimes \phi^{31} & \gamma_5 \otimes \phi^{32} & A^3 & \gamma_5 \otimes \phi^{34} \\ \gamma_5 \otimes \phi^{41} & \gamma_5 \otimes \phi^{42} & \gamma_5 \otimes \phi^{43} & A^4 \end{pmatrix} \tag{6}$$

where

$$A^p = \sum_i a_i^p \,\partial\!\!\!/ b_i^p, \qquad p = 1, 2 \tag{7a}$$

$$A^p = \sum_i a_i^p \,\hat{\partial\!\!\!/} b_i^p, \qquad p = 3, 4 \tag{7b}$$

and

$$\phi^{pq} = \sum_i a_i^p (m_{pq} b_i^q - b_i^p m_{pq}), \qquad p \neq q \tag{8}$$

Note that the A^3 and A^4 are given in terms of the exotic differential structure. They will be the source of the $SU(3)$ part of the gauge group. The additional $U(1)$ term A^3 is the price we have to pay for the "exactness" of the $SU(3)$ gauge symmetry: noncommutative geometry prefers broken gauge symmetries. It is still an open question whether noncommutative geometry provides us with new (quantum?) symmetries; see Connes (1994), Várilly and Garcia-Bondía (1993), and Chamseddine *et al.* (1993) for details. There is one subtle step in the reduction of the gauge symmetry from $SU(2) \otimes U(1) \otimes U(1) \otimes SU(3)$ to $SU(2) \otimes U(1) \otimes SU(3)$. Namely, one should require that the $U(1)$ part of the associated connection is equal to Y and the $U(1)$ part of the $SU(3)$ connection and the "exotic" $U(1)$ factor is equal to $-Y$ [a more elegant but equivalent treatment can be found in Várilly and Garcia-Bondía (1993)]. But these are defined with respect to different differential structures! This can be done only locally, as the exotic differential structure defines a different set of smooth functions than the standard one (and vice versa). We will return to this problem later. This defines the algebraic structure of the standard model. To obtain the Lagrangian, we have to calculate the curvature Θ, $\Theta = \pi(d\rho) = \Sigma_i [D, a_i][D, b_i]$. This can be easily done. The bosonic part of the action is given by the formula

$$I_{\mathrm{YM}} = \mathrm{Tr}_\omega(\Theta^2 |D|^{-4}) \tag{9}$$

where Tr_ω is the Diximier trace defined by (Connes, 1994; Várilly and Garcia-Bondía, 1993)

$$\mathrm{Tr}_\omega(O) = \lim \frac{1}{\log N} \sum_{i=0}^{i=N} \mu_i(O) \tag{10}$$

Here μ_i denotes the ith eigenvalue of the (compact) operator O. The Diximier trace gives the logarithmic divergences, and the result is zero for operators in the ordinary trace class. We will use the heat kernel method (Gilkey, 1974, 1984a,b; Hurt, 1983; Mańka and Sładkowski, 1989). For a second-order positive pseudodifferential operator O: $L^2(E) \to L^2(E)$, where $L^2(E)$ denotes the space square-integrable functions on the vector bundle E, the operator

$$e^{-tO} = \frac{1}{2\pi i} \int_C e^{-t\zeta}(\zeta \cdot \mathrm{Id} - O)^{-1} \, d\zeta \tag{11}$$

is well defined for Re $t > 0$. Then the Mellin transformation (Gilkey, 1984a)

$$\int_0^\infty e^{-tO} t^{s-1} \, dt = \Gamma(s) O^{-s} \tag{12}$$

provides us with the formula

$$|D|^{-4} = \int_0^\infty dt \; t e^{-t|D|^2} \tag{13}$$

Now we have to restrict ourselves to the case

$$m_{31} = m_{32} = m_{41} = m_{42} = m_{13} = m_{14} = m_{23} = m_{24} = 0$$

in (2) so that the free Dirac operator takes the form

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \tag{14}$$

where D_2 is defined with respect to an exotic differential structure. This allows us to calculate the Diximier trace and the notion of a point retains its ordinary spacetime sense. This is not very restrictive, as the $SU(3)$ gauge symmetry is unbroken. Calculation of the Diximier trace in the general case is more involved (if at all possible) and we would lose the convenient spacetime interpretation. The relation (Gilkey, 1974)

$$e^{-t(D_1 \oplus D_2)} = e^{-t(D_1)} \oplus e^{-t(D_2)} \tag{15}$$

leads to the following asymptotic formula:

$$\operatorname{tr}((f \oplus \hat{f}) e^{-t|D|^2}) = \int dx^4 \sqrt{g} f \left(\frac{a_0}{t^2} + \frac{a_1}{t} + \cdots \right) + \int \hat{d}x^4 \sqrt{\hat{g}} \hat{f} \left(\frac{\hat{a}_0}{t^2} + \frac{\hat{a}_1}{t} + \cdots \right) \tag{16}$$

where a_i are the spectral coefficients (Gilkey, 1974, 1984a,b; Hurt, 1983; Mańka and Sładkowski, 1989), g is the metric tensor, dots denote the finite terms in the limit $t \to 0$, and the caret distinguishes between the standard and exotic structures. For the Dirac Laplacians $|D_i|^2$, $i = 1, 2$, we have $a_0 = 1$ and a_1 is equal to the curvature R. This gives the following value of the Yang–Mills (bosonic) action (roughly speaking, this is the "logarithmic divergence" term):

$$I_{\mathrm{YM}} = \frac{1}{4} \int dx^4 \sqrt{g}\, \mathrm{TR}(\pi^2(\theta)) + \frac{1}{4} \int \hat{d}x^4 \sqrt{\hat{g}}\, \mathrm{TR}(\pi^2(\hat{\theta})) \tag{17}$$

where the trace TR is taken over the Clifford algebra and the matrix structure. As before, the caret is used to distinguish the "exotic" part of the curvature from the "nonexotic" one. Note that due to continuity, the two integrals do not feel the different differential structures, so, formally, the action looks the same as in the ordinary case. Now, standard algebraic calculations [after elimination of spurious degrees of freedom by hand (Connes, 1994; Chamseddine *et al.*, 1993; Sładkowski, 1994b) or by going to the quotient space (Várilly and Garcia-Bondía, 1993)] lead to the following Lagrangian (in the Minkowski space):

$$\begin{aligned} L_{\mathrm{YM}} = \int \{ \sqrt{g}\, \{ & \tfrac{1}{4} N_g (F^1_{\mu\nu} F^{1\mu\nu} + F^2_{\mu\nu} F^{2\mu\nu}) \\ & + \tfrac{1}{2} \mathrm{Tr}(mm^\dagger) | \partial\phi + A_1\phi - \phi^\dagger A_2 |^2 \\ & - \tfrac{1}{2} [\mathrm{Tr}(mm^\dagger)^2 - (\mathrm{Tr}\, mm^\dagger)^2](\phi\phi^\dagger - 1)^2 \} \\ & + \sqrt{\hat{g}}\, \tfrac{1}{4} N_g (F^c_{\mu\nu} F^{c\mu\nu}) \}\, d^4V \end{aligned} \tag{18}$$

The $SU(3)_c$ stress tensor $F^c_{\mu\nu}$ is smooth with respect to the exotic differential structure. We will not need the concrete values of the traces in (18), so we will not quote them [they are analogous to those in Kastler and Schücker (1992) and Garcia-Bondía (n.d.)]. There are some subtleties in the formula (18). We would be tempted to rewrite it in the orthodox form:

$$\begin{aligned} L_{\mathrm{YM}} = \int \sqrt{g}\, \{ & \tfrac{1}{4} N_g (F^1_{\mu\nu} F^{1\mu\nu} + F^2_{\mu\nu} F^{2\mu\nu} + F^c_{\mu\nu} F^{c\mu\nu}) \\ & + \tfrac{1}{2} \mathrm{Tr}(mm^\dagger) | \partial\phi + A_1\phi - \phi^\dagger A_2 |^2 \\ & - \tfrac{1}{2} [\mathrm{Tr}(mm^\dagger)^2 - (\mathrm{Tr}\, mm^\dagger)^2](\phi\phi^\dagger - 1)^2 \}\, d^4x \end{aligned} \tag{19}$$

Unfortunately, in the general case there is no relation between g and $\hat{g}$. The $\hat{g}$ may not be differentiable with respect to the standard differential structure on the spacetime manifold, so we have to present additional arguments justifying (19) (see below). Another problem is connected with the existence of two, in general different metrics g and $\hat{g}$. The metric on the whole spacetime (i.e., the two copies of S) is given by the formula (Connes, 1983)

$$d(p, q) = \sup\{ |f(p) - f(q)|; f \in A, \|[D, f]\| \leq 1\}$$

We have (Connes, 1994):

Proposition 1. 1. The restriction of the metric d on $S_1 \cup S_2$ to each copy (S_1 or S_2) is the Riemannian geodesic distance of $S_1 \cup S_2$.

2. For each point $p_i \in S_i$, the distance $d(p_i, S_{k\neq i}) = \inf\{d(p_i, q); q \in S_k\}$ is equal to λ^{-1}, $\lambda = \|M\|$, and is attained at a unique point of S_k. Here M denotes the mass part of the Dirac operator:

$$D = \begin{pmatrix} \partial\!\!\!/ \otimes \mathrm{Id} & \gamma_5 \otimes M^{\dagger} \\ \gamma_5 \otimes M & \hat{\partial\!\!\!/} \otimes \mathrm{Id} \end{pmatrix}$$

Of course, Proposition 1 is true only if the Gromov distance between the two Riemannian metrics is smaller than λ^{-1}. This gives a lower bound on the value of $\|M\|$ which for a finite-dimensional matrix $\|M\|$ is equal to its largest eigenvalue. As M describes masses of the matter fields, this may suggest the existence of additional heavy families (Sładkowski, 1994a) or the see-saw mechanism for light neutrinos. Unfortunately, our knowledge of exotic manifolds is too poor to give physical predictions.

Fermion fields are added in the usual way (Connes, 1994; Várilly and Garcia-Bondía, 1993; Chamseddine *et al.*, 1993; Sładkowski, 1994b):

$$L_f = \langle \psi | D + \pi(\rho) | \psi \rangle$$
$$= \int (\bar{\psi}_L D\!\!\!\!/\, \psi_L + \bar{\psi}_R D\!\!\!\!/\, \psi_R + \bar{\psi}_L \phi \otimes m \psi_R + \bar{\psi}_R \phi^{\dagger} \otimes m^{\dagger} \psi_L)\, d^4x \quad (20)$$

where we have included the $\pi(\rho)$ term in $D\!\!\!\!/$ (as in the ordinary covariant derivative). The quark fields are defined with respect to the exotic differential structure. Here again we might encounter the consistency problem. To proceed, let us review some results concerning exotic differential structures on $\mathbf{R}^4$ (Gompf, 1993; Brans, 1994).

An exotic $\mathbf{R}^4_\Theta$ consists of a set of points which can be globally continuously identified with the set of four coordinates (x^1, x^2, x^3, x^4). These coordinates may be smooth locally, but they cannot be globally continued as smooth functions and no diffeomorphic image of an exotic $\mathbf{R}^4_\Theta$ can be given such global coordinates in a smooth way. There are uncountable many different $\mathbf{R}^4_\Theta$. Brans (1994) proved the following theorem:

Theorem 1. There exist smooth manifolds which are homeomorphic but not diffeomorphic to $\mathbf{R}^4$ and for which the global coordinates (t, x, y, z) are smooth for $x^2 + y^2 + z^2 \geq a^2 > 0$, but not globally. Smooth metrics exist for which the boundary of this region is timelike, so that the exoticness is spatially confined.

He also conjectured that such localized exoticness can act as a source for some externally regular field, just as matter or a wormhole can. Of course, there are also $\mathbf{R}^4_\Theta$ whose exoticness cannot be localized. They might have important cosmological consequences. We also have (Brans, 1994):

Theorem 2. If M is a smooth, connected 4-manifold and S is a closed submanifold for which $H^4(M, S, \mathbf{Z}) = 0$, then any smooth, time-orientable Lorentz metric defined over S can be smoothly continued to all of M.

Now we are prepared to analyze the Lagrangian given by (18). Despite the fact that it looks like an ordinary one, we should remember that the strongly interacting fields are defined with respect to an exotic differential structure. This means that, in general, these fields may not be smooth with respect to the standard differential structure, although they are smooth solutions with respect to the exotic one. They certainly are continuous. In the noncommutative geometry approach to particle physics the spacetime manifold emerges due to interactions. This can be seen in the following way. The trace theorem relates the Diximier trace with the residue of a pseudodifferential operator on a manifold (Connes, 1994; Várilly and Garcia-Bondía, 1993). As a result we get the formula (17). We see that the two "copies" of the spacetime are topologically equivalent (homeomorphic). So the notion of a spacetime point is well defined, but we might encounter difficulties while trying to define globally some smooth structures or physical fields. In general, only those "exotic" fields that vanish outside a compact set (not necessary containing the exotic region/regions) can be expected to be differentiable with respect to the standard differential structure and consistent with the derivation of the Lagrangian (18). This is because manifolds are locally Euclidean and constant functions are differentiable; they must vanish outside a compact set because the configuration has infinite energy otherwise. Theorem 2 suggests that it might be possible to continue a Lorentz structure to all of spacetime so that (18) make sense [e.g., for a noncompact manifold M, submanifolds S for which $H^3(S; \mathbf{Z}) = 0$ satisfy the required conditions (Brans, 1994)].

The fact that a smooth metric defined over a submanifold can be smoothly continued to all of the spacetime manifold does not mean that we get the same result for any differential structure. This means that, in general, the transition from (18) to (19) requires an additional consistency condition. Again the obvious and simplest solution demands that the exotic sector can be only locally different from zero (the considered fields are continuous, and changing their values at some points does not help). In this way may "annihilate" the possible differences between $\hat{g}$ and g. This means that we can consider (19) as sort of "smooth approximation" to the description of fundamental interactions with a not explicitly written additional condition. This is compatible with experiments if strongly interacting particles and fields are exotic in the above sense. This means that strongly interacting fields probably must vanish outside a compact set to be consistent with the standard (?) differential structure that governs the electroweak sector. One can say that the exotic geometry confines strongly interacting particles to live inside baglike structures. Unfortunately, the estimation of the size of such an object is not possible without (presently unavailable) information on the global structure of exotic manifolds. *A priori*, they may be as small as baryons or as big as a quark star. What is important is the fact that such objects are not black-hole-like ones. It is possible to "get inside such an object and go back." There is no topological obstruction that can prevent us from entering the exotic region: everything is smooth, but some fields must have compact supports. One may investigate its structure as one does in the case of baryons via electroweak interactions.

It is unlikely that the above phenomenon explains confinement (one has to explain why such objects are small and so abundant), but one may wonder if such objects have astrophysical significance (it may happen that we live in such a nonexotic part of spacetime).

Of course, the above analysis is classical: we do not know how to quantize models that noncommutative geometry provides us with. Let us conclude by saying that exotic differential structures over spacetime may play an important role in particle physics. They may provide us with "confining forces" of pure geometrical origin: one does not have to introduce additional scalar fields to obtain baglike models.

We have discussed only exotic versions of $\mathbf{R}^4$, but there are also other exotic 4-manifolds. (It is likely that every 4-manifold has its exotic companions.) The proposed model is probably far from being a realistic one, but it is the only one ever constructed. We have connected the geometrical exoticness with strong interactions. We can give only one reason for doing so. Connes' construction provides us with spontaneously broken gauge symmetries. Exact gauge symmetries are "out of the way," so we have made the $SU(3)_{\mathrm{color}}$ sector "spatially exotic." Obviously, the topic deserves further investigation. One of the most important questions is, How do exotic differential structures influence quantum theory? This is under investigation.

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REFERENCES

Brans, C. H. (1994). *Classical and Quantum Gravity,* **11**, 1785.
Brans, C. H., and Randall, D. (1993). *General Relativity and Gravitation,* **25**, 205.
Chamseddine, A. H., Felder, G., and Fröhlich, J. (1992). *Physics Letters B,* **296**, 109.
Chamseddine, A. H., Felder, G., and Fröhlich, J. (1993). *Nuclear Physics B,* **395**, 672.
Connes, A. (1983). *Publications Mathematiques IHES,* **62**, 44.
Connes, A. (1988). *Communications in Mathematical Physics,* **117**, 673.
Connes, A. (1990). in *The Interface of Mathematics and Physics,* D. Quillen, G. Segal, and S. Tsou, eds., Clarendon Press, Oxford.
Connes, A. (1994). *Non-Commutative Geometry,* Academic Press, New York.
Connes, A., and Lott, J. (1990). *Nuclear Physics B Proceedings Supplement,* **18B**, 29.
DeMichelis, S., and Freedman, M. (1992). *Journal of Differential Geometry,* **35**, 219.
Donaldson, S. K. (1983). *Journal of Differential Geometry,* **18**, 279.
Freedman, M. (1982). *Journal of Differential Geometry,* **17**, 357.
Garcia-Bondía, J. M. (n.d.). Preprint [hep-th/940475].
Gilkey, P. B. (1974). *Inventiones Mathematicae,* **26**, 231.
Gilkey, P. B. (1984a). *The Index Theorem and the Heat Equation,* Princeton University Press, Princeton, New Jersey.
Gilkey, P. B. (1984b). *Invariance Theory, the Heat Equation, and the Atiyah–Singer Index Theorem,* Publish or Perish, Wilmington, Delaware.
Gompf, R. E. (1983). *Journal of Differential Geometry,* **18**, 317.
Gompf, R. E. (1993). *Journal of Differential Geometry,* **37**, 199.
Hurt, N. (1983). *Geometric Quantization in Action,* Reidel, Dordrecht.
Kastler, D., and Schücker, T. (1992). *Teoreticheskaya i Matematicheskaya Fizika,* **92**, 223 [English translation, *Theoretical and Mathematical Physics,* **1993**, 1075].
Mańka, R., and Sładkowski, J. (1989). *Physics Letters B,* **224**, 97.
Sładkowski, J. (1994a). *International Journal of Theoretical Physics,* **33**, 2381.
Sładkowski, J. (1994b). *Acta Physica Polonica B,* **25**, 1255.
Sładkowski, J. (1996). *Acta Physica Polonica B,* **27**, 649.
Várilly, J. G., and Garcia-Bondía, J. M. (1993). *Journal of Geometry and Physics,* **12**, 223.